Fourier Analysis

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Review:

Let $P_r(x)$ be the Poisson kernel on the circle, i.e.

$$P_r(x) = \frac{1-r^2}{1-2r\cos x + r^2}, \quad 0 \leq r < 1, \quad x \in [-\pi, \pi].$$

Let f be integrable on the circle, and set

$$u(r,\theta) := P_r * f(\theta), \quad 0 \leq r < 1, \quad \theta \in [-\pi, \pi].$$

Then

- $\Delta u = 0$ on $\{(r,\theta) : 0 \leq r < 1\}$
- $u(r,\theta) \xrightarrow{r \to 1} f(\theta)$ if f is cts at θ

Chap 3. Convergence of Fourier Series.

1. Mean convergence of Fourier series

We are going to prove the following results.

Thm 1.1: Let f be integrable on the circle. Then

(1) $$\int_{-\pi}^{\pi} | f(x) - S_N f(x) |^2 dx \to 0 \quad \text{as } N \to \infty$$

(L^2- convergence)

(2) (Parseval indentity)

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} | f(x) |^2 dx = \sum_{n=-\infty}^{\infty} | \hat{f}(n) |^2$$

Let $\mathcal{R}$ denote the collection of all $\mathbb{C}$-valued integrable functions on the circle. Then $\mathcal{R}$ is a linear space over $\mathbb{C}$.

- If $f \in \mathcal{R}$ and $c \in \mathbb{C}$, then $c \cdot f \in \mathcal{R}$.
- If $f, g \in \mathcal{R}$, then $f + g \in \mathcal{R}$.

Let $f, g \in \mathcal{R}$. Define

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx$$

Indeed, $\langle \cdot, \cdot \rangle$ is an inner product over $\mathbb{C}$, which satisfies the following 3 properties.

① (Conjugate symmetry)

$$\langle f, g \rangle = \overline{\langle g, f \rangle}$$

② (Linearity in the first argument)

$$\langle \alpha f + \beta g, h \rangle = \alpha \langle f, h \rangle + \beta \langle g, h \rangle$$

$\alpha, \beta \in \mathbb{C}, \quad f, g, h \in \mathcal{R}.$

(conjugate linearity in the second argument)

$$\langle f, \alpha g + \beta h \rangle = \bar{\alpha} \langle f, g \rangle + \bar{\beta} \langle f, h \rangle.$$

③ $\langle f, f \rangle \geq 0$ (positive-definite)

- Def. We say that $f, g \in \mathcal{R}$ are orthogonal if

$$\langle f, g \rangle = 0$$

In this case, we write $f \perp g$.

- Def (norm) For $f \in \mathcal{R}$, set

$$\| f \| = \sqrt{\langle f, f \rangle}$$

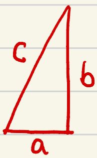
$$= \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx}.$$

we call $\|f\|$ the norm of f (or the L^2-norm of f).

Lem 1.2 ① (Pythagorean Thm)

If $f \perp g$, then

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2.$$



② (Cauchy-Schwartz inequality)

$$|\langle f, g\rangle| \leq \|f\| \cdot \|g\|.$$

③ (triangle inequality)

$$\|f+g\| \leq \|f\| + \|g\|.$$

Pf. Refer to any textbook on linear algebra. $\blacksquare$

For $n \in \mathbb{Z}$, define

$$e_n(x) := e^{inx}, \quad x \in [-\pi, \pi]$$

Then $e_n \in \mathcal{R}$.

Observe that for $f \in \mathcal{R}$,

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$= \langle f, e_n \rangle$$

Notice that $\{e_n\}_{n=-\infty}^{\infty} \subset \mathcal{R}$, it is orthonormal in the following sense:

① $\|e_n\| = 1$ for all $n \in \mathbb{Z}$

② $e_n \perp e_m$ if $n \neq m$.

Lem 1.3. For any $N \in \mathbb{N}$ and any $-N \leq n \leq N$,

$$\langle f - S_N f, e_n \rangle = 0$$

for all $f \in \mathcal{R}$.

Pf. Notice that

$$\begin{aligned}\langle f - S_N f, e_n \rangle &= \langle f, e_n \rangle - \langle S_N f, e_n \rangle \\ &= \hat{f}(n) - \widehat{(S_N f)}(n) \\ &= \hat{f}(n) - \hat{f}(n) \\ &= 0\end{aligned}$$

$\blacksquare$

Corollary 1.4. Let $f \in \mathcal{R}$, $N \in \mathbb{N}$, and let

$$P(x) = \sum_{n=-N}^{N} c_n e^{inx}$$

Then $P \perp (f - S_N f)$.

Lem 1.5. Let $P(x) = \sum_{n=-N}^{N} c_n e^{inx} = \sum_{n=-N}^{N} c_n e_n$

Then

$$\|P\|^2 = \sum_{n=-N}^{N} |c_n|^2$$

Consequently

$$\|P\| = 0 \iff c_n = 0 \text{ for } -N \leq n \leq N$$

Pf.

$$\begin{aligned}
\|P\|^2 &= \langle P, P \rangle \\
&= \left\langle \sum_{n=-N}^{N} c_n e_n, \sum_{m=-N}^{N} c_m e_m \right\rangle \\
&= \sum_{n=-N}^{N} \sum_{m=-N}^{N} c_n \overline{c_m} \langle e_n, e_m \rangle \\
&= \sum_{n=-N}^{N} |c_n|^2 .
\end{aligned}$$

$\blacksquare$

Lem 1.6 (Best approximation Thm)

Let $f \in \mathcal{R}$ and $N \in \mathbb{N}$. Then

(1)

$$\| f - S_N f \| \leq \| f - \sum_{n=-N}^{N} c_n e_n \|$$

for any $(c_n)_{-N}^{N} \subset \mathbb{C}$.

(2) The "=" holds if and only if $c_n = \hat{f}(n)$ for all $-N \leq n \leq N$.

Pf. $f - \sum_{n=-N}^{N} c_n e_n$

$$= (f - S_N f) + (S_N f - \sum_{n=-N}^{N} c_n e_n)$$

$$= (f - S_N f) + \underbrace{\sum_{n=-N}^{N} (\hat{f}(n) - c_n) e_n}_{=: P}$$

Notice that $P \perp (f - S_N f)$ (by Cor 1.4).

By Pythagorean Thm,

$$\| f - \sum_{-N}^{N} c_n e_n \|^2 = \| f - S_N f \|^2 + \| P \|^2$$

$$= \| f - S_N f \|^2 + \sum_{-N}^{N} |\hat{f}(n) - c_n|^2 .$$

Hence $\| f - \sum_{-N}^{N} c_n e_n \| \geq \| f - S_N f \|$

and "$=$" holds $\Longleftrightarrow$ $\hat{f}(n) = c_n$ for $-N \leq n \leq N$.

$\boxed{//}$

Thm 1.1. Let $f \in \mathcal{R}$. Then

① $\int_{-\pi}^{\pi} | f(x) - S_N f(x) |^2 dx \to 0$ as $N \to \infty$.

② $\| f \|^2 = \sum_{-\infty}^{\infty} |\hat{f}(n)|^2$.

Pf. We first prove ①, which is equivalent to

$\| f - S_N f \| \to 0$ as $N \to \infty$.

We claim that for any $\varepsilon > 0$, $\exists$ a trigonometric polynomial P such that

$$\| f - P \| < \varepsilon .$$

We first assume that f is cts on the circle.

In this case, by the Weirestrass approximation Thm, $\exists$ a trigonometric poly P such that

$$|f(x) - P(x)| < \varepsilon \quad \text{for all } x \in [-\pi, \pi].$$

Then

$$\|f - p\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - P(x)|^2 dx$$

$$\leq \varepsilon^2$$

Hence $\|f - p\| < \varepsilon$.

Next we consider the general case when $f \in \mathcal{R}$.

In such case, we can find a cts function g on the circle such that

- $\sup_{x \in [-\pi, \pi]} |g(x)| \leq \sup_{x \in [-\pi, \pi]} |f(x)| =: \|f\|_\infty$

- $\int_{-\pi}^{\pi} |f(x) - g(x)| dx \leq \frac{\varepsilon^2}{4 \cdot \|f\|_\infty}$.

Then $$\|f-g\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)-g(x)|^2\,dx$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\|f\|_\infty \cdot |f(x)-g(x)|\,dx$$

$$\leq \frac{\|f\|_\infty}{\pi}\int_{-\pi}^{\pi} |f(x)-g(x)|\,dx$$

$$\leq \frac{\|f\|_\infty}{\pi} \cdot \frac{\varepsilon^2}{4\cdot\|f\|_\infty}$$

$$= \frac{\varepsilon^2}{4\pi} < \frac{\varepsilon^2}{4}$$

Hence $\|f-g\| < \frac{\varepsilon}{2}$.

Then we can find a trigonometric poly p such that

$$\|g-p\| < \varepsilon/2$$

By the triangle inequality

$$\|f-p\| \leq \|f-g\| + \|g-p\|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This completes the proof of the claim.

Let P be given as in the claim.

Write $M = \deg(P)$.

Next we show that

$$\| f - S_N f \| < \varepsilon \quad \text{for all } N \geq M. \quad (*)$$

Since $N \geq M$, by the Best approximation Thm

$$\| f - S_N f \| \leq \| f - P \| < \varepsilon.$$

That is, $(*)$ holds. Therefore

$$\lim_{N \to \infty} \| f - S_N f \| = 0.$$

Now we prove (2) : $\| f \|^2 = \sum_{n=-\infty}^{\infty} | \hat{f}(n) |^2$.

Notice that $f = (f - S_N f) + S_N f$

But $S_N f \perp (f - S_N f)$. (by Cor 1.4).

By Pythagorean Thm,

$$\|f\|^2 = \|f - S_N f\|^2 + \|S_N f\|^2$$

$$= \|f - S_N f\|^2 + \sum_{n=-N}^{N} |\hat{f}(n)|^2$$

Letting $N \to \infty$ and observing that

$$\|f - S_N f\| \to 0,$$

hence

$$\|f\|^2 = \lim_{N\to\infty} \sum_{n=-N}^{N} |\hat{f}(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

$\square$

Remark: In Thm 1.1, $f \in \mathcal{R}$ can be replaced by any integrable function on $[-\pi, \pi]$.

Example 1.7. Let $f(x) = x$ on $[-\pi, \pi]$.

$$f(x) \sim \sum_{n \neq 0} \frac{(-1)^n i}{n} e^{inx}$$

By the Parseval identity, we have

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$$

$$= \sum_{n \neq 0} \frac{1}{n^2} = 2 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$$

But

$$\|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx$$

$$= \frac{1}{2\pi} \cdot \left. \frac{x^3}{3} \right|_{-\pi}^{\pi} = \frac{\pi^2}{3}.$$

So we have

$$\frac{\pi^2}{3} = 2 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \quad \text{(Euler's identity)}.$$